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Derived from semi-classical quantum field theory in curved spacetime, Unruh effect was known as a quantum effect. We find that there does exist a classical correspondence of this effect in electrodynamics. The thermal nature of the vacuum in correlation function for the uniformly accelerated detector is coming from the non-linear relationship between the proper time and the propagating length of the electromagnetic wave. Both the Coulomb field of the detector itself and the radiation supporting the detector's uniformly accelerating motion contribute to the non-vanishing vacuum energy. From this observation we conclude that Unruh temperature experienced by a uniformly accelerated classical electron has no additional effects to Born's solution for laboratory observers far away from the classical electron.

A uniformly accelerated particle moving in Minkowski vacuum is claimed to see a thermal bath, with temperature proportional to its proper acceleration  $a$ . This was first derived by Davies from the knowledge of quantum field theory in curved spacetime, while a model with a uniformly accelerated point-like quantum detector was established by Unruh [1]. Unruh effect was thought of as a pure quantum phenomenon, until Boyer [2] illustrated the classical version of it: adding random phases in the mode expansion of classical fields then average them out can give similar effects.

In classical electrodynamics, the uniformly accelerated charge(UAC) had been an confusing problem for a long time, though it is impossible to prepare any perfect experiment of this kind in a laboratory. The solution of the electromagnetic(EM) field associated with UAC was first given by Born in 1909 [3]. In his solution the magnetic field vanishes at  $t = 0$  hypersurface, hence it was claimed that there is no wave-zone in this system [4]. Half a century after, Bondi and Gold [5] gave a more general solution satisfying Maxwell equations in the whole spacetime. Then Fulton and Rohrlich [6] found that, actually, the radiating power flux does not vanish on the future lightcone; rather, it is  $2e^2a^2/3$  where  $e$  is the electric charge of the testing particle. Finally, Boulware [7] understood that the UAC not only radiates but also absorbs EM power. To keep the charge in its constant acceleration, there has to have a power-input assigned in the boundary condition at the past null infinity [8].

The motion of a charged particle in EM field is described by the Lorentz-Dirac equation [9],

$$ma^\mu = \frac{e}{c}F_{\text{in}}^{\mu\nu}v^\nu + F_{\text{ext}}^\mu + \Gamma^\mu, \quad (1)$$

where  $m$ ,  $e$ ,  $\tau$  denote the mass, charge and the proper time for the particle, respectively,  $v^\mu \equiv dx^\mu/d\tau$ ,  $a^\mu \equiv dv^\mu/d\tau$ ,  $F_{\text{ext}}^\mu$  is the non-EM force, and

$$\Gamma^\mu \equiv (F_{\text{ret}}^{\mu\nu} - F_{\text{adv}}^{\mu\nu})v_\nu = \frac{2e^2}{3c^3} \left( \dot{a}^\mu - \frac{1}{c^2}a^\nu a_\nu v^\mu \right) \quad (2)$$

is the difference of the radiation from the absorption of the particle. It is clear that, while the radiating power is measured at future null infinity globally, the field-strength differences influencing the particle motion are measured locally.

For a classical charge in uniform acceleration, the difference between the retarded and advanced field-strengths vanishes, *i.e.*,  $\Gamma^\mu = 0$ . This can be interpreted as the particle emits and absorbs photons in the same rate. Similarly, the Unruh effect from semi-classical field theory states that a uniformly accelerated detector *in equilibrium* with a thermal bath not only absorbs and counts the photons, but also emits photons in the same rate. This observation motivates the following study on the correspondence of the Unruh effect in classical electron theory.

The trajetory of a uniformly accelerated charge in Minkowski space with proper accerleration  $a$  in  $z$ -direction is a hyperbola in  $t - z$  plane, namely,

$$x^\mu = ((ac)^{-1} \sinh a\tau, a^{-1} \cosh a\tau, 0, 0), \quad (3)$$

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where  $\tau$  is the proper time in the accelerated charge's coordinate.<sup>1</sup> The total field strength of the EM field in a half of the spacetime is given by [3] [5]

$$E_z = F^{tz} = -\frac{4e}{a^2\xi^3} (a^{-2} + c^2t^2 - z^2 + \rho^2) \theta(z + ct), \quad (4)$$

$$E_\rho = F^{t\rho} = \frac{8e\rho z}{a^2\xi^3} \theta(z + ct) + \frac{2e\rho}{\rho^2 + a^{-2}} \delta(z + ct), \quad (5)$$

$$B_\phi = F^{z\rho} = \frac{8e\rho t}{a^2\xi^3} \theta(z + ct) - \frac{2e\rho}{\rho^2 + a^{-2}} \delta(z + ct), \quad (6)$$

where

$$\xi \equiv \sqrt{4a^{-2}\rho^2 + (a^{-2} + c^2t^2 - z^2 - \rho^2)^2}, \quad (7)$$

and the function  $\theta(x)$  is defined by

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}. \quad (8)$$

Note that  $F^{\mu\nu}$  is not analytic at  $z + t = 0$ . Suppose an observer comoving with the accelerated charge has the trajectory<sup>2</sup>

$$x^\mu = ((ac)^{-1} \sinh a\tau, a^{-1} \cosh a\tau, \rho, 0), \quad (9)$$

with the same  $t$  and  $z$  as the charge. Then the classical energy density measured by it is

$$\begin{aligned} \mathcal{E} &\equiv 4\pi T_{\mu\nu}(\tau) v^\mu(\tau) v^\nu(\tau) = \frac{e^2}{2\rho^4} \left(1 + \frac{a^2\rho^2}{4}\right)^{-2} \\ &= \frac{e^2}{2} \left(\frac{1}{\rho^4} - \frac{a^2}{2\rho^2} + \frac{3a^4}{16} + O(\rho^2)\right), \end{aligned} \quad (10)$$

where the stress-energy tensor for EM field is

$$T^{\mu\nu} = -\frac{1}{4\pi} \left( F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right), \quad (11)$$

and  $v^\mu(\tau)$  is the four-velocity of the charge at its proper time  $\tau$ . When the observer gets closer to the accelerated charge, *i.e.*  $\rho \rightarrow 0$ , there are two singular terms present in above small- $\rho$  expansion. The  $\rho^{-4}$ -term corresponds to the static Coulomb energy of the charge, and the  $\rho^{-2}$ -term corresponds to the radiation energy.<sup>3</sup> Interesting enough, the third term is non-vanishing as  $\rho \rightarrow 0$ .

As a conservative quantity,  $\mathcal{E}$  is a constant of the proper time  $\tau$ . Its dependence on time-like variables could be introduced as follows. By virtue of the Lorentz invariance of the system, it suffices to study the EM field at the time slice  $t = 0$  without loss of generality. When  $t = 0$  ( $\tau = 0$ ), at  $\rho$ , the total field strength is the retarded field strength starts at the proper time  $\tau_- = -\Delta/2$  and the advanced field strength ends at  $\tau_+ = +\Delta/2$ . This gives a correlation

$$\rho = \sqrt{\left(\frac{1}{a} \sinh \frac{a}{2} \Delta\right)^2 - \left(\frac{1}{a} \cosh \frac{a}{2} \Delta - \frac{1}{a}\right)^2} = \frac{2}{a} \sinh \frac{a}{4} \Delta \quad (12)$$

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<sup>1</sup>In this letter, we use the cylindrical coordinate  $d\tau^2 = c^2 dt^2 - dz^2 - d\rho^2 - \rho^2 d\phi^2$ .

<sup>2</sup>The location of the observer is chosen such that the inverse Fourier transformation below exists.

<sup>3</sup>Here the accelerated charge has  $T_{\mu\nu} u^\mu v^\nu = 0$ , where  $u_\mu$  is any spacelike vector orthogonal to  $v^\mu$ . This means that the net power-flow is zero, rather than there is no radiation from this uniformly accelerated charge. The radiated power is simply balanced by the absorbed power. Hence the radiation energy is non-vanishing for the accelerated charge.

between  $\rho$  and the parameter  $\Delta$  (see FIG.1). It should be emphasized here that  $F_{\mu\nu}(\Delta)$  are the field values at  $t, z = 0$  and the  $\rho$  in Eq.(12), rather than the field values at the position of the point charge with  $\tau = \pm\Delta/2$ . Actually,  $\Delta$  is not a measurable for the apparatus in laboratories.

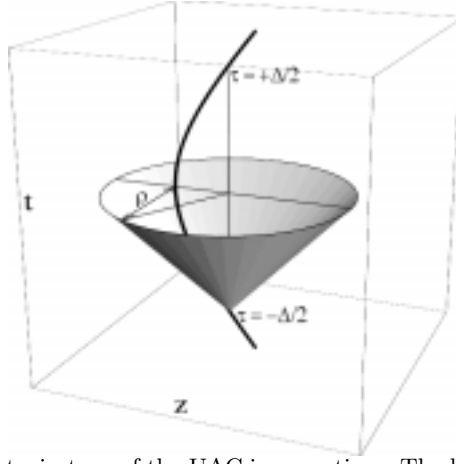


FIG. 1. The bold line is the hyperbolic trajectory of the UAC in spacetime. The lightcone starts from the UAC at its proper time  $-\Delta/2$ , and reaches the two-sphere of the lightfront at  $t = 0$  hyper-surface. The latter with radius  $a^{-1}\sinh(a\Delta/2)$  is represented by a circle in this figure.

When  $\rho \rightarrow 0$ ,  $\Delta \rightarrow 0$  also, and we recover the energy density represented in the well-known correlation function [2](up to a numerical factor) as

$$\begin{aligned}\mathcal{E} &= \lim_{\Delta \rightarrow 0} \frac{e^2 a^4}{2 \sinh^4(a\Delta/2)} = \lim_{\Delta \rightarrow 0} \frac{e^2}{2} \langle E_i^{\text{ret}}(-\Delta/2) E_{\text{adv}}^i(\Delta/2) \rangle \\ &= \lim_{\Delta \rightarrow 0} \frac{e^2}{2} \left( \frac{16}{\Delta^4} - \frac{8a^2}{3\Delta^2} + \frac{11}{45}a^4 + O(\Delta^2) \right).\end{aligned}\quad (13)$$

Again, one finds that a non-zero “vacuum energy”,  $11e^2a^4/90$ , survives after the  $\Delta^{-4}$  and  $\Delta^{-2}$  singularities (corresponding to static and radiation fields) are subtracted from Eq.(13). Note that “vacuum” does not mean that EM field vanish in space. As Born’s solution appears, the radiation of the UAC has infinite wave-length, which corresponds to static fields [10].

The singularities in  $f(\Delta) = [\sinh(a\Delta/2)]^{-4}$  can also be removed as follows. First we perform a Fourier transform

$$\tilde{f}(k) = \int d\Delta e^{ik\Delta} f(\Delta + i\epsilon), \quad (14)$$

where  $\epsilon$  is a small positive number put by hand to avoid the singularity. Let the contour to be the one surrounding the upper(lower) complex  $\Delta$ -plane for positive(negative)  $k$ . Since  $f(\Delta)$  has periodic singularities at  $\Delta = i2n\pi/a$  ( $n \in \mathbb{Z}$ ), above integration becomes ( $\omega \equiv |k|/c$ )

$$\begin{aligned}\tilde{f}(\omega) &= \sum_n e^{-2n\pi kc/a} \oint_{C_n} d\Delta \frac{e^{ik\Delta}}{\sinh^4(a\Delta/2)} \\ &= \frac{\pi}{3} \frac{1}{e^{2\pi\omega c/a} - 1} \left( \frac{c\omega}{a^2} + \frac{(c\omega)^3}{a^4} \right),\end{aligned}\quad (15)$$

where  $\sum_n \equiv \sum_{n=1}^{\infty}$  for  $k > 0$ , and  $\sum_n \equiv \sum_{n=-\infty}^0$  for  $k \leq 0$ . Then the renormalized  $\mathcal{E}$  is given by an inverse transform

$$\begin{aligned}\mathcal{E}_{\text{Ren}} &= \lim_{\Delta \rightarrow 0} \frac{e^2 a^4}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik\Delta} \tilde{f}(\omega) \\ &= \frac{8e^2 a^4}{3} \left[ \Gamma(2)\zeta(2) \left( \frac{k_B T c}{\hbar a} \right)^2 + \Gamma(4)\zeta(4) \left( \frac{k_B T c}{\hbar a} \right)^4 \right] \\ &= \frac{11}{90} e^2 a^4,\end{aligned}\quad (16)$$

which is exactly the  $O(\Delta^0)$  term in Eq.(13).

Here the well-known thermal character with temperature  $T = \hbar a / 2\pi c k_B$  arises if we identify the  $\tilde{f}(\omega)$  in Eq.(15) to Planckian-like spectrum. Note that  $\tilde{f}(\omega)$  is different from the ones for massless scalar field by a numerical factor as well as an  $\omega^1$  term, such that  $\tilde{f}(\omega)$  for EM field looks different from the ones for isotropic (3 + 1) dimensional finite temperature systems. However, above calculation shows that the temperature is simply a dummy parameter in the mode integration for the renormalized energy. If one perform a scale transformation  $\Delta \rightarrow b\Delta$ , the temperature in power spectrum (15) becomes  $b\hbar a / 2\pi$ , while the vacuum energy after integration ( $\sim \Delta^0$ ) is still the same. Both the temperature and the  $\hbar$  in temperature are obtained simply by extracting Planck factor from Eq.(15), which is not necessary in classical electrodynamics.

Hence we may say that the Unruh effect for a UAC is essentially a part of the electrodynamics. A non-zero “vacuum energy” with thermal(Planckian or non-Planckian) spectrum does not imply that the classical UAC really experiences a thermal background. Actually no additional Brownian motion for UAC is needed in classical framework, so we can get rid of the ill-defined thermal equilibrium for a *single particle* charge. What a detector in a laboratory far away from the classical charge measured are exactly those the Born’s (or Bondi and Gold’s) solution Eq.(4)-(6) describes, namely, asymptotic Larmor radiation [12]. This might explain why the predictions of some proposals given earlier [13] cannot be distinguished from the results from quantum electrodynamics.

It should be noticed that the existence of non-zero “vacuum energy” does not imply the thermal character. To obtain the Planck factor, one has to know the global property of the system ( $\Delta \in (-\infty, \infty)$ ) to make the Fourier transformation, while the “vacuum energy” is determined by local field configuration around the charge. A non-uniformly accelerating charge can also recognize the same “vacuum energy” as the ones for UAC at some instants if their accelerations are the same in that period of time.

Technically, the origin of the non-zero “vacuum energy” is the non-linear relations between the expansion variables when the acceleration is not zero. The retarded(advanced) field strength measured at  $x^\mu$  due to the charge at the point  $z^\mu(\tau)$  reads [12]

$$F_{\text{ret(adv)}}^{\mu\nu}(x) = \frac{e}{r^2 c} v^{[\mu} u^{\nu]} + \frac{e}{rc^2} \left\{ c^{-1} a^{[\mu} v^{\nu]} - u^{[\mu} \left( v^{\nu]} c^{-1} a_\alpha u^\alpha \pm a^{\nu]} \right) \right\}, \quad (17)$$

where  $A^{[\mu} B^{\nu]} \equiv A^\mu B^\nu - A^\nu B^\mu$ ,  $a^\mu$  is the four-acceleration of the charge at  $z^\mu$ , the spacelike vector  $u^\mu$  and the scalar propagating length  $r$  are defined by

$$R_{\text{ret(adv)}}^\mu \equiv x^\mu - z^\mu(\tau) = r(u^\mu \pm v^\mu/c). \quad (18)$$

While the  $r$ -expansion of  $F^{\mu\nu}$  has  $r^{-2}$  and  $r^{-1}$  terms only, the  $\Delta$ -expansion of  $F^{\mu\nu}$  for the observer in trajectory (9) has higher order terms because here  $r^{-1} = a / \sinh(a\Delta/2)$  is non-linear. Note that both the static part ( $r^{-2}$ -term) and the radiation part ( $r^{-1}$ -term) contribute to the “vacuum energy” in this case.

To conclude, we have another interesting point of view as a remark: the detector and the field should be considered as a whole, and this problem is a boundary condition issue [12]. Reversing the direction of deduction, we may say that the particle recognizes a constant non-zero “vacuum energy” or field strength at its position because we force the particle on the track of a hyperbolic motion by choosing  $F_{\text{out}}^{\mu\nu} - F_{\text{in}}^{\mu\nu} = F_{\text{ret}}^{\mu\nu} - F_{\text{adv}}^{\mu\nu} = 0$  around the charge. This corresponds to some boundary conditions at infinity. The incoming radiation from past infinity associated with this particular choice of boundary conditions serves a support to keep the detector on the hyperbolic trajectory while it’s energy dissipates by radiation. One has, of course, the freedom to choose other boundary conditions which yield non-uniform accelerations with radiation damping. Nevertheless, whether there exists such cases depends on the existences of proper solutions satisfying Maxwell equations as well as Lorentz-Dirac equations in these particular boundary conditions.

Above viewpoints can be applied to the black hole radiation. One interprets the point-like detector in Unruh’s model sees a “thermal energy” simply because the correlation function or the renormalized energy has a Planckian-like spectrum for some variables when boundary conditions were chosen. This suggests that, for a black hole in a pure gravity system, one has to choose some particular boundary conditions for field equations as well as the equation of motion for the detector to keep the detector in a rest (hence non-inertial or accelerating) frame relative to the black hole. Then the information about Hawking radiation was encoded in the gravitational field configuration near the detector, if this static solution with respect to the clock of the detector exists. However, if there does not exist proper solution for any boundary condition, then the Hawking radiation of the black hole is physically meaningless for gravitational detectors of this type.

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